coupling:

$$S_{31} = C(1+\rho)^2(1+\rho^2(T^2-C^2))/d$$
$$= jC(1+\rho)^2(1+\rho^2\epsilon^{j2\alpha})\epsilon^{j\alpha}/d$$

leakage:

$$S_{41} = 2\rho CT(1+\rho)^2/d$$
$$= j2\rho CT(1+\rho)^2\epsilon^{j2\alpha}/d$$

where

$$d = 1 - 2\rho^2(T^2+C^2) + \rho^4(T^2-C^2)^2$$
$$= 1 - 2\rho^2(1-2C^2)\epsilon^{j2\alpha} + \rho^4\epsilon^{j4\alpha}.$$

It is seen from these equations that if $\rho = 0$ the parameters of an ideal coupler are returned with $S_{11} = S_{41} = 0$, $S_{21} = T$, and $S_{31} = C$. Furthermore, interchanging T and C is seen to interchange S_{21} and S_{31}. The model also shows that for loose coupling and small imperfections, i.e., $C^2 \ll 1$ and $\rho^2 \ll 1$, if the coupling region is a quarter-wavelength long giving $\alpha \approx \pi/2$ and $T^2 \approx -1$, then the input reflection $S_{11} \approx 0$, as mentioned previously for interference-type couplers.

This demonstrates that the simple flowgraph model of a practical directional coupler of Fig. 1(b) may be used to derive the mathematical relationships describing the circuit, and the flowgraph may be used as a general-purpose building block to insert directional couplers into larger microwave networks to be analyzed by computerized flowgraph reduction programs (see footnote 2).

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REFERENCES

[1] N. Munro and P. D. McMorran, "Signal-flow-graph reduction, Mason's rule and the system matrix," *Electron. Lett.*, vol. 6, pp. 752–754, Nov. 1970.
[2] R. W. Beatty and G. H. Fentress, "An attenuation and phase shift divider circuit," *Proc. IEEE* (Lett.), vol. 56, pp. 2063–2064, Nov. 1968.
[3] W. K. Kahn and R. L. Kyhl, "The directional coupler core of an arbitrary, lossless, reciprocal 4-port," *Proc. IRE* (Corresp.), vol. 49, pp. 1699–1700, Nov. 1961.

Short Papers

On the Existence Range of the S Parameters of a Passive Two-Port Network

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Abstract—**The existence range of the S parameters of a passive two-port is analyzed and various limitations on their moduli are deduced. In particular, it is shown that when three moduli are given, the last, in general, may be upper and lower bounded by passivity conditions.**

I. INTRODUCTION

In the last three years some considerations on the bounds of the VSWR of a passive two-port network have appeared in the literature [1]–[3]. The authors dealt with the same subject some years ago, investigating the existence range of the S parameters of a passive two-port network. In particular, various limitations on the S parameters were derived. The results were used to obtain some relations between the passivity and absolute stability conditions and to determine the bounds of the input VSWR of a loaded passive two-port [4]. Furthermore, the limits of the nonreciprocity of a passive two-port and the best loss condition for an isolator were analyzed [5].

In this short paper some general limitations for the S parameters of a passive two-port network are deduced in both the nonreciprocal and the reciprocal case.

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In particular, it is shown that when all but one of the parameters are known the modulus of the last parameter can be, in general, upper and lower bounded by passivity conditions.

II. PASSIVITY CONDITIONS

For the sake of simplicity, only the case of lossy two-port networks will be considered; in that case the power entering the network is always positive and the Hermitian form giving this power as a function of the incident waves is definite positive.

The discussion of the general case of a nonnegative definite Hermitian form, including lossless networks and conditionally lossy networks (i.e., networks whose entering power can vanish for some particular excitations), does not modify the conclusions, whereas it requires many involved specifications. This case is discussed in detail in [4].

For a lossy nonreciprocal network it must be

$$1 - |S_{11}|^2 - |S_{21}|^2 > 0 \tag{1}$$

$$(1-|S_{11}|^2)(1-|S_{22}|^2) + (1-|S_{12}|^2)(1-|S_{21}|^2) - 1 - 2|S_{11}||S_{22}||S_{12}||S_{21}|\cos\alpha > 0 \tag{2}$$

where

$$\alpha = \arg S_{11} + \arg S_{22} - \arg S_{12} - \arg S_{21}. \tag{3}$$

As is well known, conditions (1) and (2) mean, respectively, that the first element and the determinant of the dissipation matrix $(\mathbf{E} - \tilde{\mathbf{S}}^*\mathbf{S})$ are positive. When these conditions hold, also the norms of the second column and of the two rows of the S-matrix are smaller than one.

From conditions (1) and (2) various limitations imposed by passivity on the S-matrix elements can be deduced, as, for example, the limits on an unmeasured reflection or transmission coefficient

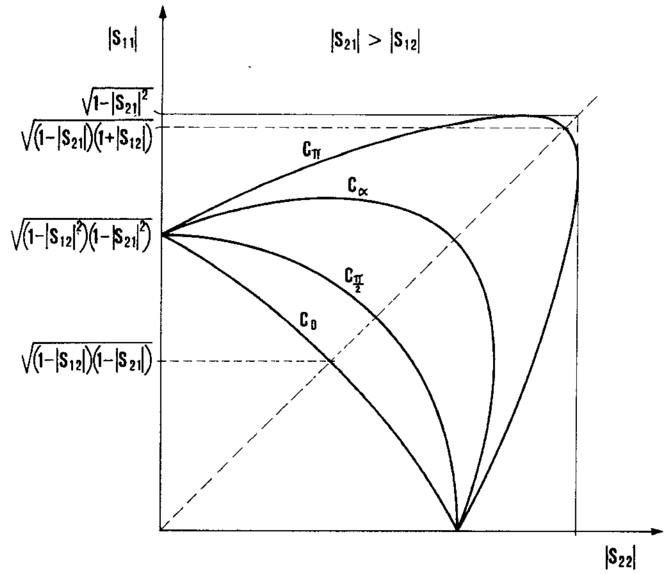

Fig. 1. The variation range of $|S_{12}|$ and $|S_{21}|$ in the $|S_{11}|$, $|S_{22}|$ plane. For a given value of $\alpha = \arg S_{11} + \arg S_{22} - \arg S_{12} - \arg S_{21}$, the region allowed by passivity conditions is that bounded by the curve C_α and the axes. The curve C_π ($\alpha = \pi$) delimits the region of minimum realizability conditions on the moduli of the S parameters and C_0 ($\alpha = 0$) the region of passivity conditions for any value of the arguments.

[4], [6], or the range of variation of the input reflection coefficient of a loaded passive two-port [4].

Another case was discussed in [5], where the authors studied the hypothetical case of a nonreciprocal two-port measured, as if it was reciprocal, by a classical input reflection coefficient measurement method as, for instance, the three-point method or Deschamps method. In this case, from $|S_{11}|$, $|S_{22}|$, $|S_{12}S_{21}|$, and α experimentally obtained, the range of variation of the ratio $|S_{12}/S_{21}|$, allowed by the passivity conditions, can be deduced.

It is possible to give a mapping of dissipation conditions (1) and (2), either in a transmission coefficient plane or in a driving-point reflection coefficient plane, thus finding a region where the S parameters of a passive network can lie.

Fig. 1 shows the allowed region in the $|S_{11}|$, $|S_{22}|$ plane; for given values of $|S_{12}|$, $|S_{21}|$, and α, $|S_{11}|$ and $|S_{22}|$ are constrained to lie into the region bounded by the axes and the curve C_α, which is a branch of the curve obtained by equating to zero the left side of condition (2). The figure is drawn for $|S_{21}| > |S_{12}|$.

Because of the symmetry properties of condition (2), the diagram of Fig. 1 is still valid exchanging $|S_{11}|$ with $|S_{22}|$ and/or reflection with transmission coefficients.

From Fig. 1 it can be deduced that the constraints deriving from condition (2) are more restrictive than those obtained by making the norms of the rows and the columns of the S-matrix smaller than one.

In particular, when $\cos\alpha \geq 0$, $|S_{11}|$ (or $|S_{22}|$) must be smaller than $[(1 - |S_{12}|^2)(1 - |S_{21}|^2)]^{1/2}$, while, when $\cos\alpha < 0$, $|S_{11}|$ (or $|S_{22}|$) must be smaller than the value of the coordinate of the vertical (or horizontal) line tangent to C_α.

III. REALIZABILITY CONDITIONS ON THE MODULI OF THE S PARAMETERS

A. General Case

Generally, one is mainly interested in the bounds on the moduli of the S parameters regardless of the value of the arguments.

The allowable regions for the moduli of the S parameters are illustrated in Fig. 1. In particular, when three moduli and α are known, the last modulus is always upper bounded and in some cases it can be also lower bounded. For example, suppose that $|S_{12}|$, $|S_{21}|$, $|S_{22}|$, and α are given; then $|S_{11}|$ has always an upper limit, given by the following expression:

$$|S_{11}| < \frac{-|S_{22}||S_{12}||S_{21}|\cos\alpha + [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2) - |S_{22}|^2|S_{12}|^2|S_{21}|^2(1 - \cos^2\alpha)]^{1/2}}{1 - |S_{22}|^2}. \quad (4)$$

Moreover, when it is $|S_{22}| > [(1 - |S_{12}|^2)(1 - |S_{21}|^2)]^{1/2}$ and $\pi/2 < \alpha \leq \pi$, $|S_{11}|$ is also lower bounded by

$$|S_{11}| > \frac{-|S_{22}||S_{12}||S_{21}|\cos\alpha - [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2) - |S_{22}|^2|S_{12}|^2|S_{21}|^2(1 - \cos^2\alpha)]^{1/2}}{1 - |S_{22}|^2}. \quad (5)$$

When it is $0 \leq \alpha \leq \pi/2$, or $|S_{22}| \leq [(1 - |S_{12}|^2)(1 - |S_{21}|^2)]^{12}$ and $\pi/2 < \alpha \leq \pi$, the lower bound is zero.

All the above formulas, as has already been noted in dealing with Fig. 1, are still valid exchanging $|S_{11}|$ with $|S_{22}|$ or transmission with reflection parameters.

B. Minimum Realizability Conditions

Condition (2) may be written as

$$\cos\alpha < \frac{(1 - |S_{11}|^2)(1 - |S_{22}|^2) + (1 - |S_{12}|^2)(1 - |S_{21}|^2) - 1}{2|S_{11}||S_{22}||S_{12}||S_{21}|} = L. \quad (6)$$

From relation (6) it is seen that the minimum realizability condition on the moduli of the S parameters is $L > -1$. When four positive real numbers satisfy this latter condition they can surely be considered as the moduli of the S-matrix elements of a passive two-port.

Observe that, if the moduli, and so L, are given, the values of the arguments must satisfy condition (6). Clearly, only when $L > 1$ there is not any limitation on the values of the arguments. This occurs when in Fig. 1 the representative point lies in the region bounded by the axes and the curve C_0, corresponding to $\alpha = 0$ or $L = 1$. Otherwise, when this point lies in the region bounded by C_0 and C_π ($\alpha = \pi$ or $L = -1$), it is $-1 < L < 1$ and the values of the arguments cannot be arbitrary but must satisfy the condition $\cos\alpha < L$.

In the case of the minimum realizability conditions, $|S_{11}|$ and $|S_{22}|$ must lie in the region bounded by the axes and the curve C_π. It can be noted that $|S_{11}|$ (or $|S_{22}|$) has the same upper bound as that deriving from condition (1), but this limit corresponds only to the value of $|S_{22}|$ (or $|S_{11}|$) given by

$$|S_{22}| = (|S_{12}/S_{21}|)(1 - |S_{21}|^2)^{1/2}.$$

If three moduli are given, e.g., $|S_{22}|$, $|S_{12}|$, and $|S_{21}|$, the bounds imposed by passivity on the last, $|S_{11}|$, are the following:

$$\frac{|S_{22}||S_{12}||S_{21}| - [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2} < |S_{11}| < \frac{|S_{22}||S_{12}||S_{21}| + [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2}. \quad (7)$$

The lower bound in condition (7) is greater than zero only when

$$|S_{22}| > [(1 - |S_{12}|^2)(1 - |S_{21}|^2)]^{1/2} \quad (8)$$

as also clearly results from Fig. 1.

C. Passivity Conditions for Any Value of the Arguments

When the values of the moduli are such that condition $L > 1$ is satisfied the arguments of the S parameters can assume every value. In Fig. 1 the allowed region for $|S_{11}|$ and $|S_{22}|$ is now that bounded by the axes and the curve C_0.

In particular, when $|S_{22}|$, $|S_{12}|$, and $|S_{21}|$ are given, in order that the passivity conditions for any value of the arguments hold, the last modulus, $|S_{11}|$, must satisfy the condition

$$|S_{11}| < \frac{-|S_{22}||S_{12}||S_{21}| + [(1 - |S_{22}|^2 - |S_{12}|^2)(1 - |S_{22}|^2 - |S_{21}|^2)]^{1/2}}{1 - |S_{22}|^2}. \quad (9)$$

IV. RECIPROCAL NETWORKS

When the two-port network is reciprocal ($S_{12} = S_{21}$), the minimum realizability condition (7) becomes

$$\frac{|S_{12}|^2}{1 - |S_{22}|} - 1 < |S_{11}| < 1 - \frac{|S_{12}|^2}{1 + |S_{22}|}. \quad (10)$$

The lower bound in condition (9) is positive only if

$$|S_{22}| > 1 - |S_{12}|^2. \quad (11)$$

Furthermore, the passivity conditions for any value of the arguments (9) reduces to

$$0 \leq |S_{11}| < 1 - \frac{|S_{12}|^2}{1 - |S_{22}|}. \quad (12)$$

The same relations hold for $|S_{22}|$, exchanging $|S_{11}|$ with $|S_{22}|$.

As regards the limits imposed on the modulus of the transmission coefficient S_{12}, when $|S_{11}|$ and $|S_{22}|$ are given they are easily found from condition (2).

In the case of the minimum passivity conditions, one obtains [6]

$$|S_{12}| < [(1 + |S_{11}|)(1 - |S_{22}|)]^{1/2} \quad (13)$$

when $|S_{11}| \leq |S_{22}|$, and

$$|S_{12}| < [(1 - |S_{11}|)(1 + |S_{22}|)]^{1/2} \quad (14)$$

when $|S_{11}| \geq |S_{22}|$.

If the passivity conditions for any value of the arguments are to be verified, it must be

$$|S_{12}| < [(1 - |S_{11}|)(1 - |S_{22}|)]^{1/2}. \quad (15)$$

The latter conditions are also directly deducible from Fig. 1, exchanging transmission with reflection coefficients and observing that, in the reciprocal case, the allowed values of $|S_{12}| = |S_{21}|$ are lying on the bisecting line.

REFERENCES

[1] A. Uhlir, Jr., "Bounds on output VSWR of a passive reciprocal two-port from forward measurements," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-18, pp. 662–663, Sept. 1970.

[2] H. J. Hindin, "A relationship between the scattering parameters of a passive lossy nonreciprocal two-port," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-19, p. 781, Sept. 1971.

[3] J. I. Glaser, "Comments on 'A relationship between the scattering parameters of a passive lossy nonreciprocal two-port,'" *IEEE Trans. Microwave Theory Tech.* (Short Papers), vol. MTT-21, pp. 135–136, Mar. 1973.

[4] E. Carli and T. Corzani, "Sulle proprietà a una frequenza della matrice di diffusione di una rete a due porte passiva," *Alta Freq.*, vol. 36, pp. 319–328, Apr. 1967.

[5] ——, "Sulla non reciprocità di una rete a due porte passiva," *Note Recensioni e Notizie*, Istituto Superiore delle Poste e Telecomunicazioni, Rome, Italy, vol. 17, pp. 792–800, Sept.–Oct. 1968.

[6] R. LaRosa and H. J. Carlin, "A general theory of wideband matching with dissipative 4-poles," *J. Math. Phys.*, vol. 33, Jan. 1955.

Multiconductor Transmission Lines and the Green's Matrix

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***Abstract*—It is shown that the study of arbitrarily terminated multiconductor transmission lines which may in general be lossy and subjected to excitation applied at an arbitrary point along the lines, may be effectively performed with the aid of the appropriate Green's matrix. The procedure is illustrated using the Chipman method of impedance measurement as well as coupled microstrip lines.**

I. INTRODUCTION

There is a considerable body of literature dealing with multiconductor transmission lines [1]–[4] which are encountered in such diverse contexts as microstrip directional couplers, overmoded waveguides, shielded pair instrumentation cables, etc., to mention but a few applications.

While known procedures are useful in dealing with various specialized cases, it will be shown that the use of the Green's matrix technique makes it possible to deal with a very wide range of situations, including excitation by voltage and current sources applied at points not necessarily coinciding with the end terminals.

In what follows we shall consider a multiconductor transmission line comprising n distinct conductors (which may be lossy) in addition to the ground path. We find that [2] in the sinusoidal steady state

$$\begin{bmatrix} V' \\ I' \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ -Y & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} + \begin{bmatrix} E_S \\ I_S \end{bmatrix} \quad (1)$$

where V, I, V', I', E_S, and I_S are n-dimensional column vectors while Z and Y are $n \times n$ square matrices. Furthermore, V, I, Z, Y, E_S, I_S, and x have the usual meaning of voltage, current, impedance per unit length, admittance per unit length, applied voltage per unit length, applied current per unit length and distance, respectively, while a prime denotes differentiation with respect to x.

II. THE MULTICONDUCTOR LINE GREEN'S MATRIX

With reference to Cole [5], a system of $2n$ differential equations

$$u' = Au + f(x) \quad (2)$$

subject to two point boundary conditions $W^a u(a) + W^b u(b) = 0$ has a solution of the form

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